MOCK FINAL EXAM - SOLUTIONS

PEYAM RYAN TABRIZIAN

Name: ______________________

Instructions: You have 3 hours to take this exam. It is meant as an opportunity for you to take a real-time practice final and to see which topics you should focus on before the actual final! Even though it counts for 0% of your grade, I will grade it and comment on it overnight, and you can pick up the graded exam tomorrow at noon in my office (830 Evans)

Note: Questions $14 - 17$ are a bit more challenging (although not impossible) than the rest! They are meant to be an extra challenge for people who finish early (and hence they are only worth 5 points each)

Note: Please check one of the following boxes:

- ☐ I will pick up my exam tomorrow between noon and 5 pm, and I want comments on my exam (Peyam/SkyDrive Tabrizian approves of this choice :))
- ☐ I will pick up my exam tomorrow between noon and 5 pm, but I don't want comments on my exam (I only want to know my score)
- ☐ I will not pick up my exam tomorrow, just grade it and enter my score on bspace!

1		15
2		10
3		20
4		10
5		15
6		15
7		10
8		10
9		15
10		10
11		20
12		15
13		15
14		5
15		5
16		5
17		5
Total		200

Date: Monday, May 9th, 2011.

1. *(15 points, 3 points each)* Evaluate the following integrals:

(a)

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \left[\sin^{-1}(x)\right]_0^1 = \sin^{-1}(1) - \sin^{-1}(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) $\int_{-1}^{1} \frac{\sin(x^3)(x^2+7x^6+1)}{\cos(x)+2} dx = \boxed{0}$ (because the function inside of the integral is odd)

(c) $\int_{-2}^{0} \sqrt{4-x^2} dx = \frac{1}{4}\pi 2^2 = \boxed{\pi}$ (the integral represents the area of the quarter circle of radius 2 in the upper-left quadrant!

(d) $\int \frac{\cos(x)}{\sin^2(x)} dx$

Let $u = \sin(x)$, then $du = \cos(x)dx$, so:

$$\int \frac{\cos(x)}{\sin^2(x)} dx = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\sin(x)} + C = -\csc(x) + C$$

(e) $\int_1^2 \frac{\ln(x)}{x} dx$

Let $u = \ln(x)$, then $du = \frac{1}{x}$, and $u(2) = \ln(2)$, and $u(1) = \ln(1) = 0$, so:

$$\int_1^2 \frac{\ln(x)}{x} dx = \int_0^{\ln(2)} u du = \left[\frac{u^2}{2}\right]_0^{\ln(2)} = \frac{(\ln(2))^2}{2}$$

2. *(10 points)*
(a) *(8 points)* Show that the function $f(x) = \cos(x) - x$ has at least one zero.

$f(0) = 1 > 0$, $f(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$, and f is continuous on $\left[0, \frac{\pi}{2}\right]$, so by the Intermediate Value Theorem, f has at least one zero (in $\left[0, \frac{\pi}{2}\right]$).

(b) *(2 points)* Using part (a), show that the function $g(x) = \sin(x) - \frac{x^2}{2}$ has at least one critical point.

Notice that $g'(x) = \cos(x) - x = f(x)$. We've shown in (a) that f has at least one zero, so g' has at least one zero, so g has at least one critical point!

3. *(20 points)* Sketch a graph of the function $f(x) = x\ln(x) - x$. Your work should include:
   - Domain
   - Intercepts
   - Symmetry
   - Asymptotes (no Slant asymptotes, though)
   - Intervals of increase/decrease/local max/min
   - Concavity and inflection points

(1) Domain: $x > 0$
(2) No $y-$intercepts, $x-$intercept $x = e$ ($f(x) = 0 \Leftrightarrow x\ln(x) - x = 0 \Leftrightarrow x\ln(x) = x \Leftrightarrow \ln(x) = 1 \Leftrightarrow x = e$)
(3) No symmetry
(4) **NO** H.A. or V.A., because:

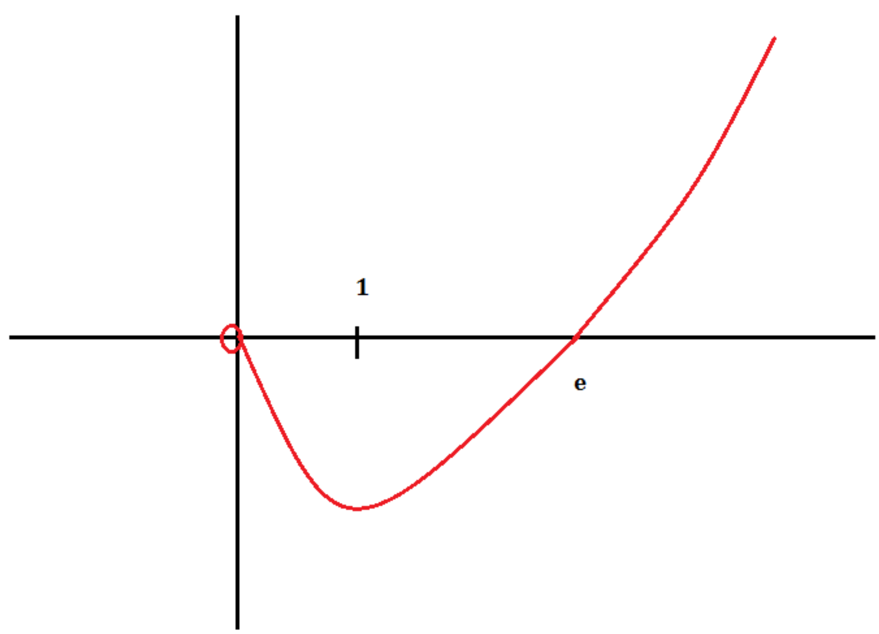
$$\lim_{x\to 0^+} x\ln(x) - x = \lim_{x\to 0^+} x(\ln(x) - 1) = \lim_{x\to 0^+} \frac{\ln(x) - 1}{\frac{1}{x}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

(H means l'Hopital's rule), and:

$$\lim_{x\to\infty} x\ln(x) - x = \lim_{x\to\infty} x(\ln(x) - 1) = \infty \times \infty = \infty$$

(5) $f'(x) = \ln(x) + 1 - 1 = \ln(x)$, f is decreasing on $(0, 1)$ and increasing on $(1, \infty)$, $(1, -1)$ is a local minimum by the first derivative test.
(6) $f''(x) = \frac{1}{x} > 0$ (if $x > 0$), f is concave up on $(0, \infty)$, no inflection points
(7) Graph:

1A/Practice Exams/Mockgraph.png



4. *(10 points)* Using the definition of the integral, evaluate $\int_0^2 \left(x^3 + x\right) dx$.
You may use the fact that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

$\Delta x = \frac{2}{n}$, $x_i = a + i\Delta x = \frac{2i}{n}$

$$\begin{aligned}
\int_0^2 x^3 + x dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left(\left(\frac{2i}{n}\right)^3 + \left(\frac{2i}{n}\right)\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \frac{16i^3}{n^4} + \sum_{i=1}^n \frac{4i}{n^2} \\
&= \lim_{n\to\infty} \frac{16}{n^4}\sum_{i=1}^n i^3 + \frac{4}{n^2}\sum_{i=1}^n i \\
&= \lim_{n\to\infty} \frac{16}{n^4}\frac{n^2(n+1)^2}{4} + \frac{4}{n^2}\frac{n(n+1)}{2} \\
&= \lim_{n\to\infty} \frac{4(n+1)^2}{n^2} + \frac{2(n+1)}{n} \\
&= 4 + 2 \\
&= 6
\end{aligned}$$

5. *(15 points, 5 points each)* Evaluate the following limits:

(a) $\lim_{x\to 0^+} \sqrt{x}\sin\left(\frac{1}{x}\right)$

$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so $-\sqrt{x} \leq \sqrt{x}\sin\left(\frac{1}{x}\right) \leq \sqrt{x}$. Now $\lim_{x\to 0^+} -\sqrt{x} = 0$ and $\lim_{x\to 0^+} \sqrt{x} = 0$, so by the **squeeze theorem** $\lim_{x\to 0^+} f(x)$ $\boxed{= 0}$

(b) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x}$

$$\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{x\to -\infty} \frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to -\infty} \frac{-x\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to -\infty} -\sqrt{1+\frac{1}{x^2}} = -1$$

Where we used the fact that $\sqrt{x^2} = |x| = -x$ (since $x < 0$ here!)

(c) $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

1) $y = \left(1+\frac{2}{x}\right)^x$

2) $\ln(y) = x\ln(1+\frac{2}{x})$

3)

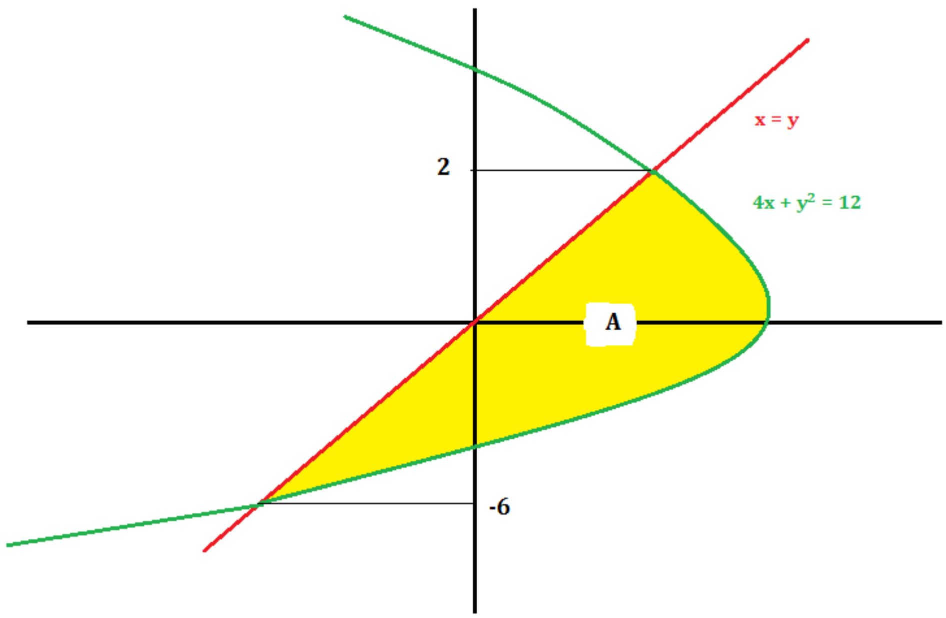
$$\begin{aligned}
\lim_{x\to\infty} \ln(y) &= \lim_{x\to\infty} x\ln(1+\frac{2}{x}) \\
&= \lim_{x\to\infty} \frac{\ln(1+\frac{2}{x})}{\frac{1}{x}} \\
&\overset{H}{=} \lim_{x\to\infty} \frac{-\frac{2}{x^2}\frac{1}{1+\frac{2}{x}}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} \\
&= 2
\end{aligned}$$

4) Hence $\lim_{x\to\infty} y = \boxed{e^2}$

6. *(15 points)* Find the area between the curves $4x + y^2 = 12$ and $x = y$

The picture is given as follows (you get it by reflecting the graph $4y + x^2 = 12$ about the line $y = x$:

1A/Practice Exams/Mockarea.png



Here, the rightmost function is $x = 3 - \frac{1}{4}y^2$, and the leftmost function is $x = y$.

Now, for the points of intersection, we need to solve:

$$\begin{aligned} 4x + x^2 &= 12 \\ (x+2)^2 &= 16 \\ x + 2 &= \pm 4 \\ x &= -6, 2 \end{aligned}$$

Hence, the area is:

$$\int_{-6}^{2} 3 - \frac{y^2}{4} - y dy = \left[3y - \frac{y^3}{12} - \frac{y^2}{2} \right]_{-6}^{2} = 6 - \frac{2}{3} - 2 + 18 + 18 - 18 = \frac{64}{3}$$

7. *(10 points)* Suppose f is an odd function and is differentiable everywhere. Prove that, for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = \frac{f(b)}{b}$ (weird question, huh? =))

Since this is a weird question, it's a Mean Value Theorem question! :)

By the Mean Value Theorem applied to $[-b, b]$, we get that, for some c in $(-b, b)$:

$$\frac{f(b) - f(-b)}{b - (-b)} = f'(c)$$

But $f(-b) = -f(b)$ since f is odd, and $b - (-b) = 2b$, so:

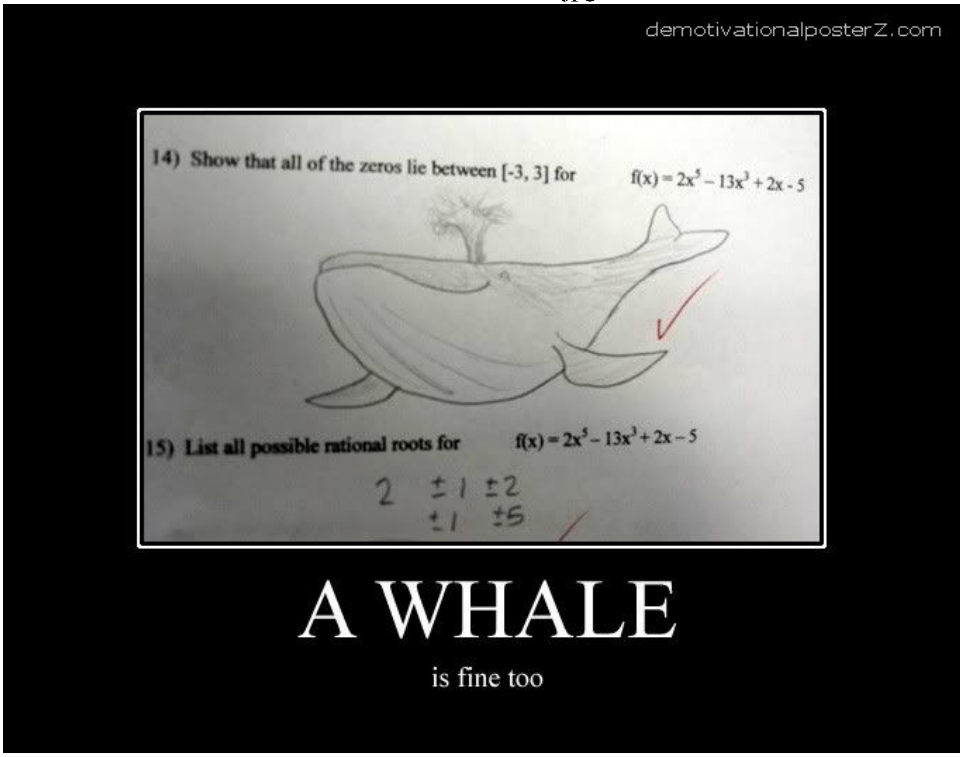
$$\frac{2f(b)}{2b} = f'(c)$$

That is:

$$\frac{f(b)}{b} = f'(c)$$

Which is what we wanted to show!

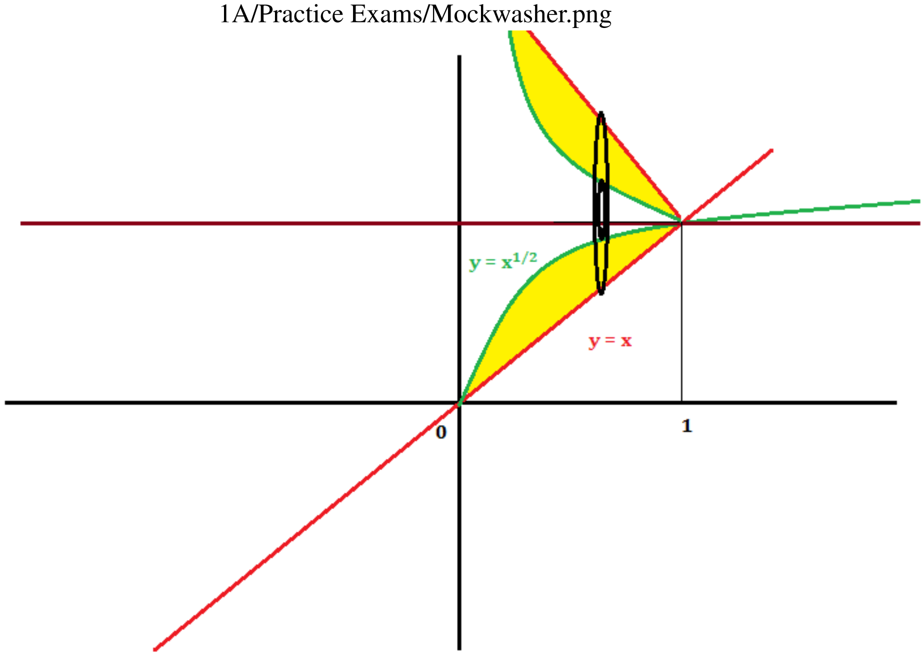
1A/Practice Exams/Whale.jpg



8. *(10 points)* Find the volume of the solid obtained by rotating the region bounded by $y = x$, $y = \sqrt{x}$ about $y = 1$

This is a typical application of the washer method! Here $k = 1$, Outer = $x - 1$, Inner = $\sqrt{x} - 1$ (see the following picture):

1A/Practice Exams/Mockwasher.png



Hence

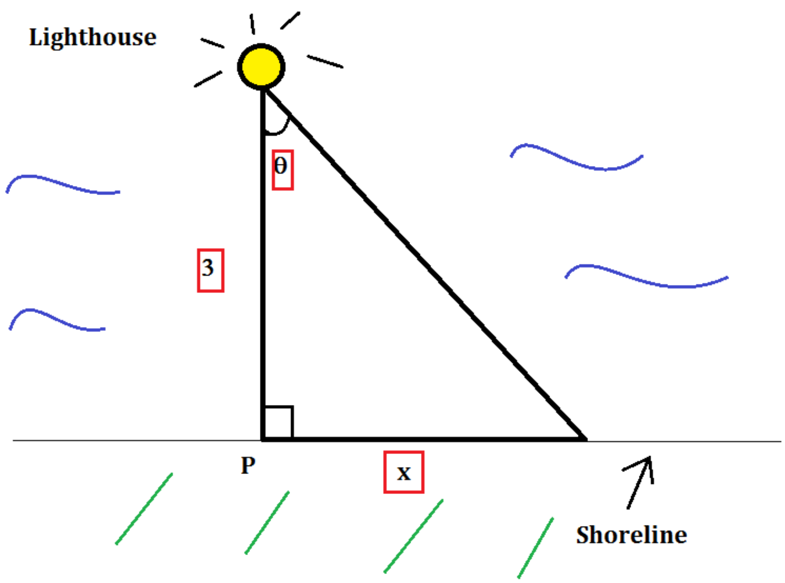
$$\begin{aligned}\int_0^1 \pi\left((x-1)^2 - (\sqrt{x}-1)^2\right) dx =& \pi \int_0^1 x^2 - 2x + 1 - x + 2\sqrt{x} - 1 dx \\ =& \pi \int_0^1 x^2 - 3x + 2\sqrt{x} dx \\ =& \pi \left[\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}}\right]_0^1 \\ =& \pi\left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3}\right) \\ =& \frac{\pi}{6}\end{aligned}$$

9. *(15 points)* A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline, and the angular velocity of the light is 8π radians per minute. How fast is the beam of light moving along the shoreline when it is 1 km away from P?

This is problem 38 in section 3.9! (in the 6th edition of the textbook)

1) Again, draw a picture of the situation:

1A/Archive/Solution Bank - Old Edition/Lighthouse.png



2) Want to find $\frac{dx}{dt}$ when $x = 1$
3)

$$\tan(\theta) = \frac{x}{3}$$

So $x = 3\tan(\theta)$
4) Hence $\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}$
5) We're given that $\frac{d\theta}{dt} = -8\pi$ (you put a minus-sign since x is decreasing!).

Moreover, by drawing the **exact** same picture as above, except with $x = 1$, we can calculate $\sec^2(\theta)$, namely:

$$\sec(\theta) = \frac{hypothenuse}{adjacent} = \frac{\sqrt{10}}{3}$$

(and the $\sqrt{10}$ we get from the Pythagorean theorem!)
It follows that $\sec^2(\theta) = \left(\frac{\sqrt{10}}{3}\right)^2 = \frac{10}{9}$.

$$\begin{aligned}\frac{dx}{dt} =& 3\sec^2(\theta)\frac{d\theta}{dt} \\ \frac{dx}{dt} =& 3(\frac{10}{9})(-8\pi) \\ \frac{dx}{dt} =& -\frac{240\pi}{9} \\ \frac{dx}{dt} =& -\frac{80\pi}{3}\end{aligned}$$

Whence $\boxed{\frac{dx}{dt} = -\frac{80\pi}{3} ft/min}$

10. *(10 points, 5 points each)* Find the derivatives of the following functions:

(a) $f(x) = \sin^{-1}(x)\sqrt{1-x^2}$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}\sqrt{1-x^2} + \sin^{-1}(x)\frac{-2x}{2\sqrt{1-x^2}} = 1 - \sin^{-1}(x)\left(\frac{x}{\sqrt{1-x^2}}\right)$$

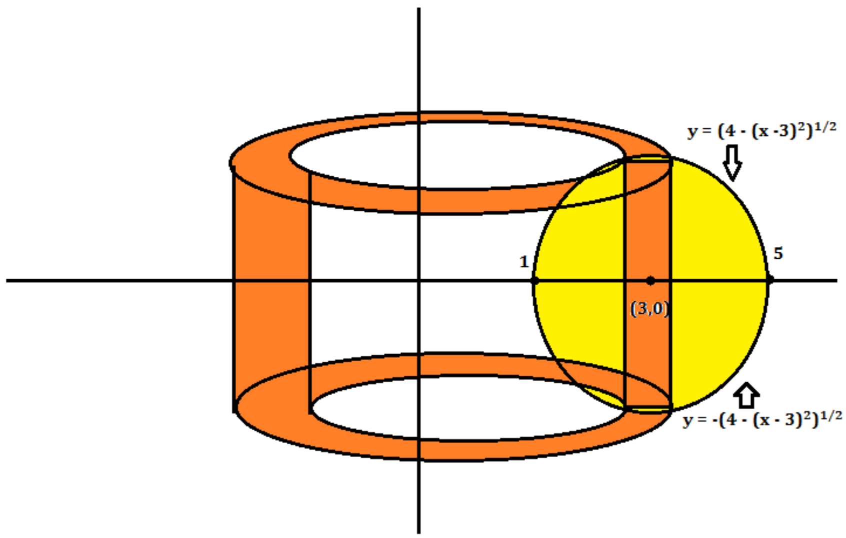
(b) $f(x) = x^{\ln(x)}$

1) $y = x^{\ln(x)}$
2) $\ln(y) = \ln(x)\ln(x) = (\ln(x))^2$
3) $\frac{y'}{y} = 2\frac{\ln(x)}{x}$
4) $y' = y\left(2\frac{\ln(x)}{x}\right) = x^{\ln(x)}\left(2\frac{\ln(x)}{x}\right)$

11. *(20 points)* Find the volume of the donut obtained by rotating the disk of center $(3,0)$ and radius 2 about the $y-$axis.

Picture:

1A/Practice Exams/Mocktorus.png



For this, let's use the shell method! $x=0$, so $k=0$, and Radius $=|x-0|=|x|=x$. Also, the equation of the circle is $(x-3)^2+y^2=4$, so $y=\pm\sqrt{4-(x-3)^2}$, and Outer $=\sqrt{4-(x-3)^2}$ and Inner $=-\sqrt{4-(x-3)^2}$, so Height = Outer - Inner = $2\sqrt{4-(x-3)^2}$.

Hence:

$$V=\int_1^5 2\pi x\left(2\sqrt{4-(x-3)^2}\right)dx=\int_1^5 4\pi x\sqrt{4-(x-3)^2}dx$$

Now let $u=x-3$, then $du=dx$, $x=u+3$, and $u(1)=-2$, $u(5)=2$, so:

$$V=\int_{-2}^{2} 4\pi(u+3)\sqrt{4-u^2}du=4\pi\int_{-2}^{2} u\sqrt{4-u^2}du+12\pi\int_{-2}^{2}\sqrt{4-u^2}du$$

However, the first integral is 0 because $u\sqrt{4-u^2}$ is an odd function, and the second integral is $\frac{1}{2}\pi 2^2=2\pi$, because it represents the area of a semicircle of radius 2!

Hence, we get:

$$V=0+12\pi(2\pi)=24\pi^2$$

12. *(15 points)* Show that the equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_0, y_0) is:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

First of all, by implicit differentiation:

$$\begin{aligned}\frac{2x}{a^2} + \frac{2yy'}{b^2} &= 0\\ y'\left(\frac{2y}{b^2}\right) &= -\frac{2x}{a^2}\\ y' &= -\frac{b^2}{a^2}\frac{2x}{2y}\\ y' &= -\frac{b^2}{a^2}\frac{x}{y}\end{aligned}$$

It follows that the tangent line to the ellipse at (x_0, y_0) has slope $-\frac{b^2}{a^2}\frac{x_0}{y_0}$, and since it goes through (x_0, y_0), its equation is:

$$y - y_0 = \left(-\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

And the rest of the problem is just a little algebra!

First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = -b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$ya^2 y_0 - a^2 (y_0)^2 = -b^2 x_0 x + b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2 y_0 + b^2 x_0 x = a^2 (y_0)^2 + b^2 (x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 + \frac{b^2}{a^2} x_0 x = (y_0)^2 + \frac{b^2}{a^2}(x_0)^2$$

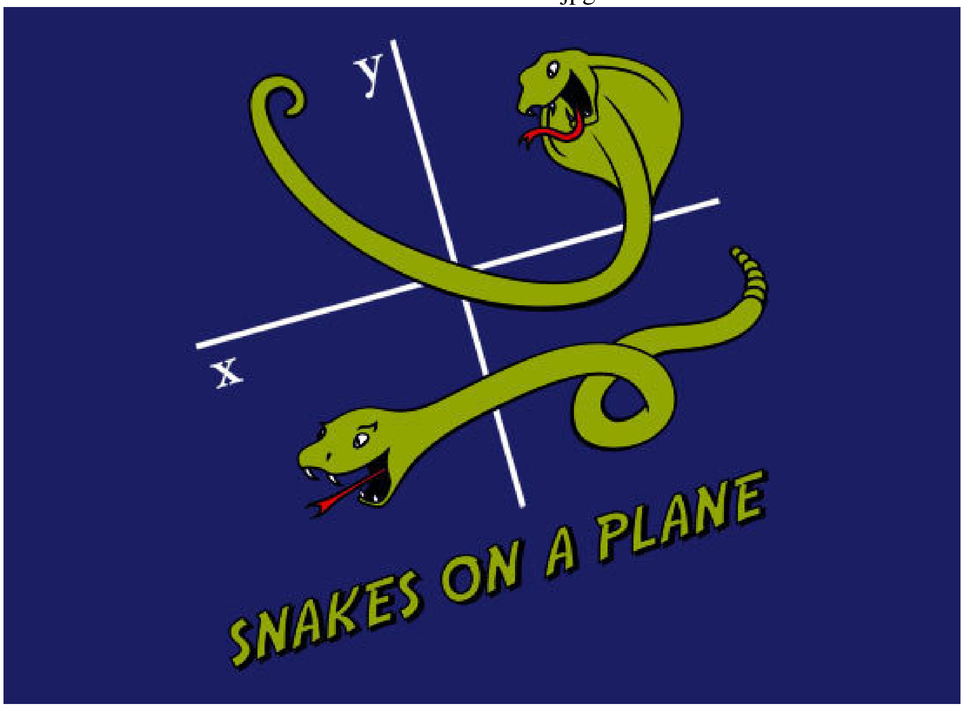
And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} + \frac{x_0 x}{a^2} = \frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the ellipse, $\frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2} = 1$, we get:

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

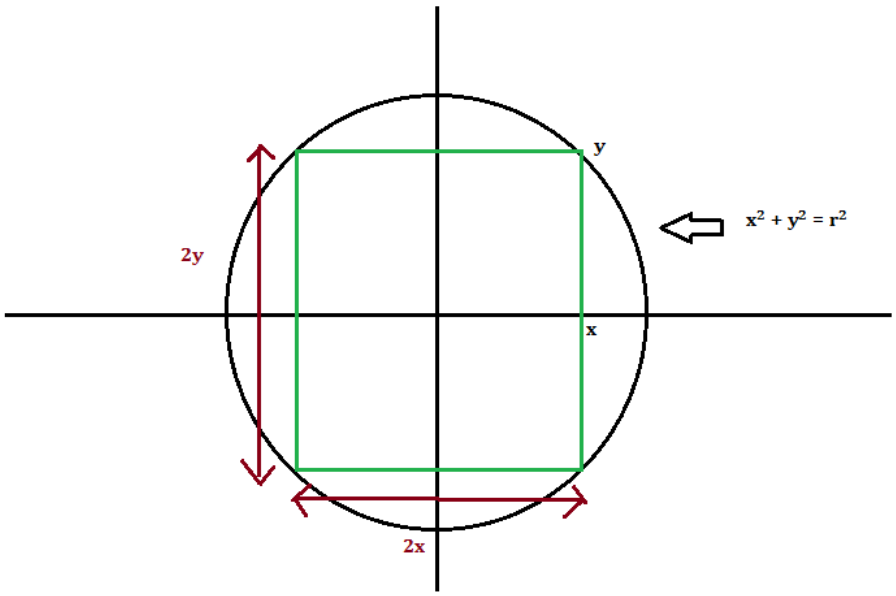
1A/Practice Exams/Snake.jpg



13. *(15 points)* Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

1) First draw a good picture!

1A/Practice Exams/Mockrectangle.png



2) Based on your picture, the length of the rectangle is $2x$ and the width is $2y$, and the area is $A = (2x)(2y) = 4xy$. But since (x, y) is on the circle, $x^2 + y^2 = r^2$, so $y = \sqrt{r^2 - x^2}$, so $\boxed{A(x) = 4x\sqrt{r^2 - x^2}}$.

3) The constraint is $0 \leq x \leq r$
4) $A'(x) = 4\sqrt{r^2 - x^2} + 4x\frac{-2x}{2\sqrt{r^2-x^2}} = \frac{4}{\sqrt{r^2-x^2}}\left(r^2 - x^2 - x^2\right) = 0 \Leftrightarrow r^2 - 2x^2 = 0 \Leftrightarrow x = \frac{r}{\sqrt{2}}$.

   Also, $A(0) = A(r) = 0$, and $A(\frac{r}{\sqrt{2}}) > 0$ (we don't really care what it is, as long as it's positive), so by the closed interval method, $x = \frac{r}{\sqrt{2}}$ is an absolute maximizer.

   So our answer is:

$$Length = 2x = \sqrt{2}r, Width = 2y = 2\sqrt{r^2 - \frac{r^2}{2}} = 2\frac{r}{\sqrt{2}} = \sqrt{2}r = x$$

   So the optimal rectangle is a **square!!!**

Note: I would like to remind you that questions $14 - 17$ are more challenging than the rest, but you can give them a try if you want to, they are not impossible to do!

14. *(5 points)* Solve the differential equation $T' = T - 5$.

Hint: Let $y = T - 5$. What differential equation does y solve?

$y' = T' = T - 5 = y$, so $y' = y$, so $y = Ce^t$, so $T - 5 = Ce^{kt}$, so $\boxed{T = 5 + Ce^t}$

15. *(5 points)* If f is continuous on $[0,1]$, show that $\int_0^1 f(x)dx$ is finite.

Since f is continuous on $[0,1]$, by the extreme value theorem, f attains an absolute maximum M and an absolute minimum m, so $m \leq f(x) \leq M$. Integrating, we get: $\int_0^1 m dx \leq \int_0^1 f(x)dx \leq \int_0^1 M dx$, so $\boxed{m \leq \int_0^1 f(x)dx \leq M}$, so $\int_0^1 f(x)dx$ is finite because both m and M are finite!

16. *(5 points)*

(a) Use l'Hopital's rule to show:

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x)$$

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &\overset{H}{=} \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &\overset{H}{=} \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2} \\ &= \frac{f''(x) + f''(x)}{2} \\ &= f''(x)\end{aligned}$$

(The tricky part is that we're differentiating with respect to h here, not with respect to x)

(b) Use (a) to answer the following question: If $f(x) = x^2 \sin(\frac{1}{x})$ with $f(0) = 0$, does $f''(0)$ exist?

The above formula with $x = 0$ gives:

$$f''(0) = \lim_{h\to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2}$$

But here $f(0) = 0$, $f(h) = h^2 \sin(\frac{1}{h})$, and $f(-h) = (-h)^2 \sin(-\frac{1}{h}) = -h^2 \sin(\frac{1}{h}) = -f(h)$. Hence:

$$f''(0) = \lim_{h\to 0} \frac{f(h) - 0 - f(h)}{h^2} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

So $\boxed{f''(0) = 0}$

17. *(5 points)* If f is differentiable (except possibly at 0) and $\lim_{x\to\infty} f(x) = 0$, is it true that $\lim_{x\to\infty} f'(x) = 0$? Prove it or give an explicit counterexample!

You might **think** this is true, but it is actually **FALSE**! Let $f(x) = \frac{\sin(x^2)}{x}$. Then f is differentiable except at 0, $\lim_{x\to\infty} f(x) = 0$ by the squeeze theorem. Moreover:

$$f'(x) = \frac{\cos(x^2)(2x)(x) - \sin(x^2)}{x^2} = 2\cos(x^2) - \frac{\sin(x^2)}{x^2}$$

And $\lim_{x\to\infty} f'(x) \neq 0$. In fact, the limit does not exist! Although $\lim_{x\to\infty} \frac{\sin(x^2)}{x^2} = 0$ by the squeeze theorem, $\lim_{x\to\infty} 2\cos(x^2)$ does not exist, which causes $\lim_{x\to\infty} f'(x)$ not to exist!

The interesting thing about this function is that although it goes to 0 at ∞, it oscillates wildly, and the oscillations are faster than rate of convergence of the function at 0 (that's why I chose the factors x^2 and x, this wouldn't work with $\frac{\sin(x)}{x}$, its oscillations are not that bad!)

You're done!!!

Any comments about this exam? (too long? too hard?)

1A/Practice Exams/Soccer.jpg